Notes on the Droop quota

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1 Introduction

STV methods have historically used one of two quotas: the Hare quota v/s (votes divided by seats) or the Droop quota $v/(s+1)$ (votes divided by seats plus one) [1, 2].

The Hare quota v/s is the largest quota such that s candidates can be elected. Methods employing the Hare quota typically deal in whole votes, and use the integer portion of the calculation: $\lfloor v/s \rfloor$.

With the Hare quota, it is possible for a majority bloc of voters to elect only a minority of seats, in particular when the number of seats is odd. The Droop quota, the smallest quota such that no more candidates can be elected than there are seats to fill, addresses this problem. Furthermore, the Hare quota is vulnerable to strategic voting and vote management, which the Droop quota makes much less likely to succeed. More generally, the Droop quota figures in the Droop proportionality criterion; thus Woodall [3]:

> The most important single property of STV is what I call the *Droop proportionality criterion* or *DPC*. Recall that if v votes are cast in an election to fill s seats, then the quantity $v/(s+1)$ is called the *Droop quota*.
>
> DPC. If, for some whole numbers *k* and *m* satisfying $0 < k \leq m$, more than k Droop quotas of voters put the same m candidates (not necessarily in the same order) as the top m candidates in their preference listings, then at least k of those m candidates should be elected. (In the event of a tie, this should be interpreted as saying that every outcome that is chosen with non-zero probability should include at least k of these m candidates.)

Nicolaus Tideman (after Michael Dummett) calls this "(*k*+1)-proportionality for solid coalitions", or (*k*+1)-PSC [2, p269].

The Droop quota, like the Hare quota, is often rounded to an integer. From O'Neill's description of the proposed BC STV rules [4]:

> The "Droop quota" will be the formula for calculating the number of votes required by a candidate for election in a district. The quota formula is:
>
> $$\left(\frac{\text{total number of valid ballots cast in the district}}{1 + \text{number of members to be elected}} \right) + 1$$
>
> Fractions are ignored.

More compactly: $\lfloor v/(s+1)+1 \rfloor$.

Henry Droop himself defined his quota as $mV/(n+1)+i$, where V voters have *m* votes each, the number of seats is n, and i is the number necessary to reach the smallest integer greater than $mV/(n+1)$ [5]. When *m* is 1, this gives the same result as $\lfloor v/(s+1)+1 \rfloor$, though differently expressed.

If *m* is 10^k, this is the equivalent of working to k decimal places with one vote each. Droop says that i rounds up to the next integer, not to the next multiple of m, making it quite clear that Droop himself would think that any such increment should be in the last decimal place used, not a whole integer. (It is unlikely, however, that Droop contemplated using $m > 1$ for STV elections.)

It seems to have been nearly a century before the purpose of the $+i$ was queried, when in the 1970s Frank Britton pointed out to Robert Newland that it was never needed except in the case of a tie for all remaining places and, if that happened, it did not help to resolve the tie. This led to the 1976 version of the ERS rules to replace the 1972 version.

In fact, Droop's quota does not satisfy his wish of being the smallest possible that cannot elect too many, unless it is insisted that the same quota has to apply to all, for once the incremented quota has been applied to the first elected, a smaller quota would be safe for all the rest. It might be argued that it would be unfair to make the first elected keep a larger number, but it is no more so than filling the last places on less than a quota, as is traditional practice.

However, there is an extra point of importance when hand counting, well explained by Robert Newland (in a letter to Bernard Black, quoted with permission in ERS Technical Committee paper TC 88/2). He wrote "in earlier days I have had Droop quotas of 2.01, 3.01, 4.01, etc. If the Droop quota was, say, 4.01, and one or more candidates had 4 votes, then one was obliged to carry out the farce of transferring votes to those candidates, and then transferring away all except 0.01 of the added votes, even though those candidates already demonstrably had sufficient votes that they must be elected. Now, since 1976, the Britton quota has avoided this nonsense".

The new ERS rules avoided "this nonsense" only for quotas that could be expressed exactly in two decimal places, but, as we shall see, the principle can be extended if we can represent quotas exactly.

2 Terminology

Some sources reserve the term "Droop quota" for the rounded-up $\lfloor v/(s+1)+1\rfloor$. Tideman calls $v/(s+1)$ the "NB quota", after Newland and Britton [2, p271], while Newland referred to it as the "Britton quota" [quoted above]. Wikipedia (as of this writing) calls $v/(s+1)$ the "Hagenbach-Bischoff quota" [6], but *Electoral System Design* glosses "Hagenbach-Bischoff Quota" as "Another term for the *Droop Quota"* [7].

A cursory survey of online literature, including *Voting matters*, suggests that the name "Droop quota" is commonly used for any quota between $\lfloor v/(s+1)+1\rfloor$ and $v/(s+1)$. The difference can be as much as a full vote, usually insignificant in large elections, but often significant in small ones.

3 Problems

The exact (unrounded) Droop quota $v/(s+1)$ has two potential problems.

Too many winners.

If the quota is exactly $v/(s+1)$, then $s+1$ candidates can receive exactly a quota. This problem can be addressed in several ways.

- Adjust the quota upward, typically by the nominal limit of computational precision, but in some rules as much as to the next higher integer.
- Use the exact quota, but elect on exceeding, rather than simply reaching, the quota [8].
- Use the exact quota. If there are $s+1$ winners, they must be tied; break the tie.
- Use the exact quota, as with the last case, but deferring the election of candidates with exactly a quota until s or fewer candidates remain. Break ties as required.

Limitations of numerical representation.

Typical implementations use binary or decimal arithmetic, in which a quota such as 100/(2+1) cannot be exactly represented. Again, there are several ways to address the problem.

- Adjust the quota upward to a value that can be represented, the limiting case being the integer quota $\lfloor v/(s+1)+1\rfloor$.
- Use the exact quota if it can be exactly represented; otherwise adjust the quota upward to the smallest representable value that is greater than the exact quota.
- Use rational arithmetic, so that all values can be represented exactly. This approach is likely to be computationally expensive, and has not to our knowledge been implemented.
- Use quasi-exact fixed-point or floating-point arithmetic with guard digits (see appendix below).

ERS97, which uses two decimal digits of precision, represents 100/(3+1) exactly (as 25.00) but rounds 100/(2+1) up (to 33.34) [9]. Integer-based methods use $\lfloor v/(s+1)+1\rfloor$, so that these two quotas become 26 and 34. OpenSTV's implementation of Meek's method uses 25.000001 and 33.333334 by default (six decimal digits of precision, always rounding up) [10]. The "Algorithm 123" implementation of Meek's method treats the underlying computational precision as exact, ignoring truncation and rounding errors, and breaks ties when too many candidates reach the quota [11].

DPC failure. STV rules such as Irish or BC STV that use a quota of $\lfloor v/(s+1)+1\rfloor$ do not satisfy the Droop proportionality criterion (DPC), as demonstrated by this example from Robert Newland [12] (two parties, four candidates per party, seven seats to be filled).

Party A:	101	101	101	98	(Total 401)
Party B:	100	100	100	99	(Total 399)

If the quota is 100 ($v/(s+1)$), Party A takes four seats, and Party B three. If the quota is 101 ($\lfloor v/(s+1)+1 \rfloor$) or, more generally, greater than $100\frac{2}{3}$, Party A takes three seats, and Party B four, a DPC violation. (The Hare quota shares this difficulty, leading to its problems with vote management.)

Premature election. Requiring that candidates reach (rather than exceed) the exact quota $v/(s+1)$ raises an additional difficulty, as in this example due to Tideman; two to be elected:

4	A
4	B
3	C D
1	D C

The quota is 4; A and B are elected. While this case does not violate Woodall's Droop proportionality criterion (since no group has *more* than one Droop quota), the solid coalition for C & D ought to carry the same weight as those for A and B, and we should discover the A–B–C tie. This problem does not arise if the rule requires that candidates exceed the exact quota, or if it defers the election of candidates with exactly a quota until all candidates with fewer votes have been excluded.

Unintended tiebreaking (1). Methods that round the quota up have a problem with this example (two to be elected):

4	A B
2	C

The exact quota is 2. If we round that quota up to 2.01, A is elected, we transfer the surplus of 1.99 to B, so that C beats B by a vote of 2 to 1.99. In our opinion, it is clear that B and C should be regarded as tied.

Unintended tiebreaking (2). In the previous example, rounding the quota up may be seen as gratuitous. In this example, rounding up serves another purpose (five to be elected):

6	A E
4	B E
7	C D F
3	D F

A, B, C & D are elected, and E & F should tie (we have two coalitions of 10 voters each). However, the exact quota of 20/6 cannot be exactly represented in either base 2 or base 10. If the quota is rounded up, E is elected because F suffers from more rounding error than E. This problem can be resolved by using a method that employs an exact quota in all cases.

Inexact representation can also lead to the appearance of a tie when there is in fact none. Suppose that, as a consequence of surplus transfers, Candidate A has 1 + 99/100 votes, and Candidate B has $1+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}$ Candidate B should beat Candidate A, but if $\frac{1}{3}$ is represented as 0.33, they will appear to be tied at 1.99.

4 Conclusion

Should we prefer one approach to another?

The $\lfloor v/(s+1)+1 \rfloor$ integer version of the Droop quota is defensible in the context of a hand-counting rule that deals with whole-vote transfers only, so that only whole numbers are involved in the count. Such rules have other problems, though, that are beyond the scope of this paper.

Methods using fractional surplus transfers should use an exact quota and require that candidates exceed the quota, or, alternatively, require that candidates reach the quota, defer the election of candidates with exactly a quota, breaking ties as required.

If exact computation is not practical, errors resulting from the deviation can be minimized by rounding up as little as possible—for example, rounding up to the nominal precision of the specified rule.

The choice of an STV method generally has more significant implications than do the details of quota calculation, and anyone who has examined the ballots in a large election will be painfully aware that clerical errors or errors due to voter carelessness (or mischief) will generally far outweigh calculation differences in the millionths of votes. Nonetheless, it may be seen as a reasonable desideratum that our calculations not introduce unnecessary errors into our results—perhaps especially in the simple examples above, and that the Droop Proportionality Criterion be strictly observed, especially when such a result may be obtained with little additional effort.

5 Appendix: Quasi-exact arithmetic with guard digits

Here we describe a method of performing quasi-exact STV calculations with fixed-point or floating-point arithmetic. The results are exact if the specified conditions are met.

Perform arithmetic to the precision $p + g$ digits, where p is the nominal computational precision and g is additional guard digits; when making comparisons, ignore differences less than half the nominal precision 10^{-p}, and display results rounded to p decimal places. For example, with a nominal precision p of 6 digits, perform computations to 10 digits ($g = 4$), and define ($a \approx b$) as ($|a-b| < 0.0000005$), where $\approx$ is read "essentially equal to" (Knuth's terminology [13]). For this method to succeed, the nominal precision p must be adequate to represent any "real" differences, and there must be sufficient guard digits g to absorb any accumulated truncation errors. This approach is available as an option in a forthcoming version of OpenSTV as well as in Lundell's Perl-based STV counter [14].

It has been observed that the relation $\approx$ as defined here is not transitive; that is, ($a \approx b$) and ($b \approx c$) do not imply ($a \approx c$). While this is true in general, the problem can be avoided by making p and g sufficiently large. Moreover, it may be considered that the loss of transitivity is more than compensated for by the fact that we avoid the embarrassing problem that (for example) $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} \neq 1$.

An alternative method is to define p and g as above, and to test for equality after rounding to p decimal places. This method preserves the transitivity of the equality relation at the expense of (potentially) treating arbitrarily close values as unequal, as long as they are on opposite sides of a rounding boundary. Again, this problem is avoided to the extent that p and g are sufficiently large. Ensuring that p and g are sufficiently large is not trivial. As Wichmann has observed [15], it is possible to create election examples in which very small surplus transfers can affect the outcome; in his example, a succession of two transfers results in a significant difference of 1/16 000 000 of a vote, and it would be straightforward to extend his example to require even more precision.

6 References

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