

Voting matters  
*for the technical issues of STV*

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## Editorial

There are 4 papers in this issue:

- I. D. Hill and C. H. E. Warren: *Meek versus Warren*.

This article compares two computer-based STV counting algorithms. Although the Meek version seems to be the only version which is widely used, readers of *Voting matters* should surely appreciate the differences and draw their own conclusions.

- I. D. Hill and Simon Gazeley: *Sequential STV — a further modification*.

This paper considers a variant of STV in which later preferences are used to exclude candidates. The modification described here has proved necessary due to two issues which are described in the paper.

- Earl Kitchener: *A new way to break STV ties in a special case*.

This short paper considers one special case in which the proposal is surely non-controversial. This is followed by summary and moderated debate on breaking ties produced by the editor with assistance from those listed.

- P Kestelman. *Apportionment and Proportionality: A Measured View*.

The author's abstract reads: Apportionment (allocating seats to multi-member constituencies equitably) can illuminate proportionality (allocating seats to parties fairly) and its quantification. Sainte-Laguë (Webster) is the fairest method of apportionment — and electoral principle. Several disproportionality measures have been proposed: among which the Loosemore-Hanby Index straightforwardly measures Party total over-representation. UK general elections (First-Past-the-Post) have clearly proved non-PR; and even nominally PR elections of British MEPs and Regional Assemblies have yielded only semi-PR ('broad PR'). Allowing for vote transferability, multimember STV in Ireland has mediated full PR (despite low District Magnitude); while Alternative Voting in Australia has arguably proved semi-PR.

## The New Zealand STV elections

A Parliamentary investigation (Justice and Electoral Committee) is under way into the delays in producing the results. It has not yet reported.

Steve Todd reported in the last issue that the ballot data should be available. In fact, the electoral officers were divided on the provision of this data so that complete data is only available for 15 of the 79 elections. (There were 81 STV elections, but two were not contested.) A table giving the availability of the data is available on <http://stv.sourceforge.net/>.

## The British Columbia Referendum for STV

The Referendum produced a majority for STV, but not the 60% to ensure that the necessary legislation will be passed. It is unclear at this stage what will happen.

*Readers are reminded that views expressed in **Voting matters** by contributors do not necessarily reflect those of the McDougall Trust or its trustees.*



# Meek *versus* Warren

I. D. Hill and C. H. E. Warren  
No email available.

## 1 Where we agree (I.D. Hill and C.H.E. Warren)

We admire traditional STV methods (Newland and Britton rules [1] and other similar methods) as being a good approximation to what STV is trying to achieve, while being easy enough to do by hand within a reasonable length of time, but in this electronic age, we ought to do better than that. Of course we accept that the ability to count by hand is an advantage; but does such an advantage justify the consequence that, quite often, the set of candidates who best meet the voters' wishes are not elected? We think not. But if we seek to campaign for something better, we need to agree on the better thing that we should support.

We agree that fairness is of prime concern in a voting system, but it is a tricky concept — one only has to listen to politicians all claiming that taxation, for example, must be fair (“and must be seen to be fair” as if that addition helped), while totally disagreeing with each other about what is fair and what is not.

The Meek method [2] and the Warren method [3] are very similar to each other but, in deciding how much of each vote is retained by an elected candidate and how much is passed on to the next choice, the Meek method uses multiplicative ‘keep values’ but the Warren method uses additive ‘portions apportioned’. We here denote the Meek keep value and the Warren portion apportioned for candidate C as  $c_m$  and  $c_w$  respectively. These quantities have a value between 0 and 1, and they are calculated so that, if a candidate has a surplus, their use reduces the vote for that candidate to just the quota. The calculation of these quantities so that they meet this requirement is a mathematical problem, usually requiring a computer. All that we need to know in this paper is that they can be calculated.

With the Meek method  $c_m$  is defined as the proportion of the vote that is passed to candidate C which candidate C retains, so that  $(1 - c_m)$  is the proportion of that vote that is passed on. In the case of a ballot that reads ABC...

the portion of vote which A retains is  $a_m$   
the portion of vote which A passes on to B is  $(1 - a_m)$

the portion of vote which B retains is  $(1 - a_m)b_m$

the portion of vote which B passes on to C is  $(1 - a_m)(1 - b_m)$

the portion of vote which C retains is  $(1 - a_m)(1 - b_m)c_m$

the portion of vote which C passes on is  $(1 - a_m)(1 - b_m)(1 - c_m)$

and so on.

From the above statements we see why the Meek keep values are called multiplicative.

With the Warren method  $c_w$  is defined as the portion of a vote that is apportioned to candidate C if such apportionment is possible. In the case of a ballot that reads ABC...

the portion of vote which is apportioned to A is  $a_w$   
if  $a_w + b_w > 1$ , the portion of vote which is apportioned to B is  $(1 - a_w)$

and nothing is apportioned to C and beyond  
if  $a_w + b_w \leq 1$ , the portion of vote which is apportioned to B is  $b_w$

if  $a_w + b_w \leq 1$  and  $a_w + b_w + c_w > 1$ ,  
the portion of vote which is apportioned to C is  $(1 - a_w - b_w)$   
and nothing is apportioned beyond

if  $a_w + b_w + c_w \leq 1$ , the portion of vote which is apportioned to C is  $c_w$

and so on.

From the above statements we see why the Warren portions apportioned are called additive.

Although a Meek keep value  $c_m$  may, in some circumstances, turn out to have the same value as a Warren portion apportioned  $c_w$ , in general their numerical values are different.

The methods are equally easy to program for a computer and, for real voting patterns as distinct from test cases, they nearly always produce the same answers, not in numerical terms but in terms of which candidates are elected and which are not. In those circumstances, we agree that it does not matter too much which is used, so it is preferable to support the one that is better in principle — but which one is that?

We recognise that impossibility theorems, such as Woodall's theorem [4], show that to seek an absolute ideal is a 'wild-goose chase'. It follows that it will always be possible to produce particular examples that tell against any given method. Unlike proving a proposition in pure mathematics, where one counter-example is enough to demonstrate that we have failed, here we always need to look at examples in a comparative sense, not an absolute sense, deciding which faults to allow for the sake of avoiding others.

## 2 Why I prefer the Meek method (I.D. Hill)

To my mind the essence of STV is this — if we have a quota of 7, and 12 identical votes putting A as first preference and B as second (with no others for A) then 7 votes must be held for A as a quota while the other 5 are passed to B and, from that point on, behave exactly as if they had originally been 5 votes for B as first preference. The fact that those voters had A as first preference, and A has been elected, has been fully allowed for in holding 7 votes back and the other 5 votes are now simply B votes.

In practice, we never get such identical votes, so the only fair way of doing things is, instead of holding 7 complete votes back and passing on 5 complete votes, to hold back  $\frac{7}{12}$  of each vote and pass on  $\frac{5}{12}$  of each vote, but the principle, that the 12 votes each of value  $\frac{5}{12}$  should together have the same power as 5 complete votes, remains the same. This principle is fulfilled by the Meek method, but not by the Warren method. Because perfection is impossible, it could be that some advantage could be shown by the Warren method that

would outweigh this disadvantage, but I am not aware that any advantage has been claimed for it that is strong enough to do so.

If, at the next stage, we have 5 votes with B as first preference, plus our 12 votes each now of value  $\frac{5}{12}$ , we have 10 votes altogether pointing at B. Only 7 are needed for a quota so  $\frac{7}{10}$  needs to be retained allowing  $\frac{3}{10}$  to be passed on, so the 5 votes are passed on with a value of  $\frac{3}{10}$ , giving them a total power of  $1\frac{1}{2}$  votes. If the 12 votes are passed on with a value of  $\frac{5}{12}$  times  $\frac{3}{10}$ , that gives them a total power of  $1\frac{1}{2}$  votes too, showing that 12 each of value  $\frac{5}{12}$  are being treated just like 5. To get that effect necessarily requires a multiplicative rule, not an additive rule.

To look at it from a slightly different angle, the rule should be that the proportions of the total vote for a candidate that come from different sources, and are used in deciding that the candidate can now be elected, should be maintained in the amounts of vote retained and transferred. Thus, in the same example, the votes from the AB voters and from the B voters that are used to decide to elect B are in proportion 1 to 1, whether the Meek or the Warren method is used. With Meek, the votes retained from the two groups are  $3\frac{1}{2}$  and  $3\frac{1}{2}$ , also 1 to 1, and those transferred are  $1\frac{1}{2}$  and  $1\frac{1}{2}$ , also 1 to 1. With Warren, the votes retained are  $4\frac{16}{17}$  and  $2\frac{1}{17}$ , or 2.4 to 1, and those transferred are  $\frac{1}{17}$  and  $2\frac{16}{17}$ , or 1 to 50, devoid of all the proportionality that I believe they should have.

The Meek method is able to promise voters that once their first  $n$  choices have all had their fates settled, either as excluded or as elected with a surplus, a fair share of their vote will be passed to their  $(n + 1)$ th choice, unless no more transfers are possible because all seats are now filled. How much is a fair share may, perhaps, be arguable (though I do not personally see it as such) but it cannot possibly be zero, which the Warren method often makes it.

Thus the basis of STV in Meek mode is that everything has to be done in proportion to the relevant numbers at the time. This means that if we have 1 ballot paper of value 1 pointing at XY, and  $n$  ballot papers each of value  $\frac{1}{n}$  pointing at XZ, and X's papers are to be redistributed, then what happens to Y and to Z from those papers should be identical.

Suppose 8 candidates for 7 seats, counted by Newland and Britton rules. If there are 40 votes reading 5 ABCG, 5 ABCH, 5 ABDG, 5 ABDH, 5 ABEG, 5 ABEH, 5 ABFG, 5 ABFH, it is evident from the symmetry that ABCDEF must be elected but the final seat is a tie between G and H. If, however, there is a 41st vote

reading BH, that ought to settle it in favour of H, but those rules declare it still to be a tie between G and H to be settled at random. Either Meek or Warren counting would have awarded the seat to H.

However, suppose the 41st vote, instead of being just BH reads BCDEFH. Again Newland and Britton rules fail to discover that the symmetry has been broken, and incorrectly call it a GH tie. But now so do Warren rules. With Meek rules, only 0.012 of the vote gets through as far as H, but that is enough to tilt the balance to get the right result.

In the past, when Hugh Warren and I have argued about this, each of us has, from time to time, put forward an example with an 'obviously right' answer which the other one's preferred method failed to find. However, with those examples, the other one of us never accepted that the answer in question was 'obviously right'. It was therefore necessary to produce something where the answer could not be denied. I claim to have done this with the example: 4 candidates for 3 seats, and just 3 votes: 1 ABC, 1 BC, 1 BD. Without even knowing anything about STV, it must be clear that ABC is a better answer than ABD. Meek does elect ABC, but Warren says that C and D tie for the third seat and a random choice must be made between them. Unless something equally convincing can be found that points the other way, that seems to me to be conclusive.

So far as I am aware, the only actual advantage claimed for Warren over Meek is that it is supposed to give consistency when some voters change the order of two candidates both of whom are elected anyway. This seems to me to be only a very slight advantage, and Warren rules do not always succeed even in that. With 5 candidates for 4 seats and votes 9 ABCD, 8 BD, 8 CE, 7 D, 7 E, either Meek or Warren elect ABCD. But if the ABCD votes had been ACBD instead, either Meek or Warren would elect ABCE.

The difference arises from the fact that one quota of votes is necessarily ineffective and changing the order of some preferences can change which votes those are and thus, in marginal cases, affect the result. I suggest that in practice any such inconsistency would never be noticed and is of very minor importance compared with making the count so that everything is kept in proportion to the numbers concerned.

I am less convinced than I was even that such behaviour can be called an anomaly. If two candidates are both elected anyway, it would seem at first sight that, if some voters change the order of those two, it ought not to affect who else gets elected, but is that really a good

rule? In this example, there is some connection between B and D, and between C and E. We do not know what the connection is, but it is clearly there since every voter putting B first puts D second, while every voter putting C first puts E second. The second choice of the A supporters is then saying what they think about the feature that gives the connection. In such circumstances, it does not seem unreasonable that if the A voters prefer B to C that helps D, but if they prefer C to B that helps E, particularly when the first preferences for D and E are tied.

Overall, while accepting that the Warren method works quite well, it does not seem to me to have any real advantage over the Meek method, and its failure to meet what I regard as basic requirements can sometimes lead to a result that I would think unfortunate. Given how wrong it seems, I am surprised that it works as well as it does.

### **3 Why I prefer the Warren method (C.H.E. Warren)**

I prefer the Warren method because I consider it to be based on a better principle.

The main principle behind the Warren method (given as the second principle in [3]) can be stated as: if a voter votes for candidates A, B, C in that order, and if candidates A and B each have a surplus of votes above the quota, then, on principle, no portion of the vote for ABC shall be credited to candidate C unless the voter has contributed, as far as he is able, the same portion of his vote to the election of candidate B as other voters who have contributed to the election of candidate B.

The main principle behind the Meek method (given as principle 2 in [2]) can be stated as: if a voter votes for candidates A, B, C in that order, and if candidates A and B each have a surplus of votes above the quota, then, on principle, a portion of the vote for ABC shall be credited to candidate C.

These different principles lead to the different rules as set out in paragraphs 3 to 8 of section 1.

I think that whether one prefers the Meek method to the Warren method, or vice versa, should be based on principle, and I prefer the principle upon which the Warren method is based. As stated in paragraph 8 of section 1, because of the impossibility theorems, it will always be possible to produce particular examples that tell against any given method. So I prefer to rest my case on the matter of principle, rather than on seeking

examples of where the Warren method gives a ‘better’ result than the Meek method. Nevertheless, an example will be given, not with the object of showing that one method gives a better result than the other, but of showing how the two methods can give different results.

Consider the following election for 3 seats by 39996 voters, for which the quota is 9999.

10000 vote ABC  
 100 vote AE  
 10000 vote BD  
 9998 vote C  
 9898 vote D

The numbers have been chosen so that, unlike the situation in real elections, the count can be done manually.

Under the Meek method the count can be portrayed as follows:

Voter	Number of such voters	Portion of vote contributed by each voter to each candidate				
		A	B	C	D	E
Keep value		0.99	0.99	1	1	1
ABC	10000	0.99	0.0099	0.0001	0	0
AE	100	0.99	0	0	0	0.01
BD	10000	0	0.99	0	0.01	0
C	9998	0	0	1	0	0
D	9898	0	0	0	1	0
Total vote for each candidate		9999	9999	9999	9998	1

Under the Warren method the count can be portrayed as follows:

Voter	Number of such voters	Portion of vote contributed by each voter to each candidate				
		A	B	C	D	E
Portion apportioned		0.99	0.9899	1	1	1
ABC	10000	0.99	0.01	0	0	0
AE	100	0.99	0	0	0	0.01
BD	10000	0	0.9899	0	0.0101	0
C	9998	0	0	1	0	0
D	9898	0	0	0	1	0
Total vote for each candidate		9999	9999	9998	9999	1

We see from these tables that the Meek method elects candidates A, B, C, whereas the Warren method elects candidates A, B, D.

We observe that the Meek and Warren methods are in agreement as to the portion of vote that each of the ABC voters and the AE voters contribute to candidate

A, which is in keeping with the Warren principle that all contributors to the election of a candidate should contribute the same portion of their vote.

We observe that the Meek and Warren methods differ in the portion of vote that each of the ABC voters, and each of the BD voters, contribute to candidate B. Both methods ask the BD voters to contribute closely 99% of their vote to candidate B, and ask the ABC voters to contribute only closely 1% to candidate B. The Warren method accepts this difference, because, although it would have preferred that all groups of voters contributed the same portion, it recognises that the ABC voters did use up all that was left of their vote after contributing to candidate A, and could not contribute more.

The Meek method is desirous that, if a voter votes for a candidate who is elected with a surplus, then that voter should not be asked to contribute so much of his vote to that candidate that he has nothing to pass on. Accordingly, although each ABC voter is contributing only closely 1% of his vote to the election of candidate B, compared with the 99% that each BD voter is contributing, Meek’s principle requires that the ABC voters shall contribute slightly less than 1% of their vote to the election of candidate B in order that a portion, which amounts to about one ten-thousandth of a vote, shall be passed to candidate C.

This shows what the difference between the Meek and Warren methods amounts to. In my opinion the difference raises the question as to whether the ABC voters, who have contributed only closely 1% of their vote to the election of candidate B, whereas the BD voters have contributed closely 99% towards the same end, merit the right, in these circumstances, to pass on a portion of their vote to candidate C, as Meek’s principle requires, at the expense of expecting the BD voters to bear even more of the burden of electing candidate B. If one thinks that the right should be afforded, then one should prefer the Meek method. But if one thinks that it would not be fair to afford this right, then one should prefer the Warren method.

#### 4 References

- [1] R.A. Newland and F.S. Britton. *How to conduct an election by the Single Transferable Vote*. 2nd edition. Electoral Reform Society. 1976.
- [2] B.L. Meek. A new approach to the Single Transferable Vote. *Voting matters*, issue 1, 1–11. 1994.

*Hill and Warren: Meek v. Warren*

- [3] C.H.E. Warren. Counting in STV elections. *Voting matters*, issue 1, 12–13. 1994.
- [4] D.R. Woodall. An impossibility theorem for electoral systems. *Discrete Mathematics*, 66, 209–211. 1987.

# Sequential STV — a further modification

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## 1 Introduction

We had hoped that our earlier paper [1] would be the final version of the Sequential STV system, but we have found two examples since then that seem to call for further amendment.

The aim is to find a system that will be noticeably like ordinary STV but: (1) will correct unfairness, if any, to candidates excluded by the reject-the-lowest rule; (2) will automatically reduce to Condorcet’s method rather than Alternative Vote when there is only a single seat.

It seeks to find a set of  $n$  candidates that observes Droop Proportionality [3], which we regard as an essential feature of any worthwhile voting system, and is preferred by the largest majority of voters to any other possible set of  $n$ . Tideman’s CPO-STV [2] has similar objectives. The successful set will usually be such that any set of  $n+1$  candidates, consisting of those  $n$  and 1 more, will result in the election of those  $n$  when an STV election is performed and in this case we refer to the successful set as a Condorcet winning set.

In a small election, or when  $n=1$ , it would be relatively easy and quick to do a complete analysis, as CPO-STV does. The challenge is to find a way that will work in a reasonable time in large elections, where such a complete analysis would be impracticable. We recognise that the meanings of ‘a reasonable time’ and ‘impracticable’ are open to dispute, and that what is practicable will change as computers continue to get faster. As Tideman and Richardson say “We are not yet at a point where computation cost can be ignored completely”.

In cases where it is practicable to do a complete analysis of all sets of  $n+1$ ,  $n+2$ , etc., it might be possible to find a solution that, in some sense, is preferable to that produced by this system that (after an initial stage)

looks only at sets of  $n+1$  and only at some of those. We think, however, that it would be hard to claim a severe injustice to any non-elected candidate after this system had been used, and it does keep things within manageable limits. It would be interesting to compare the performance of Sequential STV and CPO-STV, but this has not been done yet.

Of the two worrying examples, one showed that the system, as previously given, could fail to preserve Droop Proportionality, while the other showed that we were a little over-optimistic in claiming that, if the special procedure to deal with a Condorcet paradox had to be invoked, “most of the original candidates will be either excluded or certainities, [so] there is no need to fear an astronomical number of tests needing to be made”. This second example was highly artificial and the optimism was probably justified for any real voting pattern that is at all likely to occur, but even artificial patterns ought not to cause trouble.

To cure the first of these troubles it is necessary, when the special procedure is used, to let it exclude just one candidate before restarting the main method, instead of continuing to use the special procedure. To cure the second, the special procedure has been much simplified, to calculate a value for each continuing candidate based upon Borda scores, and to exclude the one with the lowest score. We emphasise that in real elections, as distinct from specially devised test cases, Condorcet loops rarely occur and so the special procedure is rarely called into use.

Borda scores on their own, as an electoral method, we regard as a very poor option. Those elected are far too dependent upon whether or not other (non-winning) candidates are standing, and the method is much too open to tactical voting; but as a method of helping to sort out a Condorcet paradox, they can be useful. Where a paradox arises, we know that there cannot be a good result because, whoever is elected, it is possible to point to some other option that a majority of the voters pre-

ferred; so the best that can be done is to try for a not-too-bad result and, for this limited purpose, Borda scores can serve.

## 2 Revised version of Sequential STV

All STV counts mentioned are made by Meek's method. It would be possible to use a similar system with some other version of STV but, since many counts are to be made using the same data, to try it other than by computer would make little sense. If a computer is required in any case, Meek's method is to be preferred.

An initial STV count is made of all candidates for  $n$  seats, but instead of dividing into those elected and those not elected, it classifies those who would have been elected as probables, and puts the others into a queue, in the reverse order of their exclusion in that STV count, except that the runner-up is moved to last place as it is already known that an initial challenge by that candidate will not succeed. Having found the probables and the order of the queue, further rounds each consist of  $n+1$  candidates, the  $n$  probables plus the head of the queue as challenger, for the  $n$  seats. Should a tie occur during these rounds, between a probable and a challenger, it is resolved by maintaining the current situation; that is to say, the challenger has not succeeded.

If the challenger is not successful, the probables are unchanged for the next round and the challenger moves to the end of the queue, but a successful challenger at once becomes a probable, while the beaten candidate loses probable status and is put to the end of the queue. The queue therefore changes its order as time goes on but its order always depends upon the votes.

This continues until either we get a complete run through the queue without any challenger succeeding, in which case we have a solution of the type that we are seeking, or we fall into a Condorcet-style loop.

A loop may have been found if a set that has been seen before recurs as the probables. If the queue is in the same order as before then a loop is certain and action is taken at once. If, however, a set recurs but the queue is in a different order, a second chance is given and the counting continues but, if the same set recurs yet again, a loop is assumed and action taken.

In either event the action is the same, to exclude all candidates who have never been a probable since the last restart (which means the start where no actual restart has occurred) and then to restart from the begin-

ning except that the existing probables and queue are retained instead of making a new initial STV count.

If there is no candidate who can be so excluded, then a special procedure is used, in which each continuing candidate, other than any who has always been a probable since the last restart, is classified as 'at-risk'. Taking each continuing candidate, a Borda score is calculated, as the sum over all votes of the number of continuing candidates to whom the candidate in question is preferred, taking all unmentioned continuing candidates as equal in last place. A continuing candidate who is not mentioned in a particular vote is given, for that vote, the average score that would have been attained by all those unmentioned. In practice it can help to give 2 points instead of 1 for each candidate beaten, because all scores, including any averages required, are then whole numbers.

The at-risk candidate with the lowest score (or a random choice from those with equal lowest score) is then excluded and the main method restarted from the beginning, except that the existing probables and queue order are retained instead of making the initial STV count. If the newly excluded candidate was one of the queue, he or she is merely removed from the queue, but if the candidate was a probable, the candidate at the head of the queue is reclassified as a probable and removed from the queue. Then a restart is made from the beginning except that the existing probables and queue are retained instead of making a new initial STV count.

## 3 Proof of Droop Proportionality compliance

The 'Droop proportionality criterion' says that if, for some whole numbers  $k$  and  $m$  (where  $k$  is greater than 0 and  $m$  is greater than or equal to  $k$ ), more than  $k$  Droop quotas of voters put the same  $m$  candidates (not necessarily in the same order) as their top  $m$  preferences, then at least  $k$  of those  $m$  candidates will be elected.

We know that a normal STV count is Droop Proportionality compliant so, in Sequential STV, for  $k$  and  $m$  defined as above, at least  $k$  of the  $m$  will be probables at the first count. If on a later count a challenger takes over as a probable then, because that was also the result of an STV count, there will still be at least  $k$  of the  $m$  among the probables, even if the replaced candidate was one of the  $m$ . This ensures compliance if no paradox is found.

If a paradox is found, at least  $k$  of the  $m$  will have been probables at some time since the last restart, so

excluding all who have not been probables must leave at least  $k$ . If the special procedure, using Borda scores, is required, then if only  $k$  exist,  $k$  will have always been probables since the last restart, and so are not at risk of exclusion, but if there are more than  $k$ , the exclusion of just one of them must leave at least  $k$ . This ensures compliance where a paradox is found.

## 4 Examples

### Example 1

This is the example that showed the old version of Sequential STV to fail on Droop Proportionality. With 9 candidates for 3 seats, votes are

10	ABCDEFGH	10	BCDAFGHI
10	CDABGHIE	11	DABCHIEF
19	EFGHIDAB	19	FGHIEBCD
1	GHIEFCDA		

41 votes (more than 2 quotas) have put ABCD, in some order, as their first choices so, to satisfy Droop Proportionality, at least 2 of them must be elected. The old version elected DEF but the new version elects ADE.

### Example 2

This is the example that showed the old version of Sequential STV not always to finish within a reasonable time. With 40 candidates for 9 seats, votes are

69	ABCDE	94	BCAED	98	CBAED
14	DEBAC	60	ECBDA	64	FGJHI
43	GIFJH	42	HJIGF	97	IHGJF
33	JIHGF	32	KLMNO	44	LMNOK
56	MNOKL	76	NOKLM	90	OKLMN
18	PQRST	91	QRSTP	69	RSTPQ
21	STPQR	76	TPQRS	36	UVWXY
78	VWXYU	99	WXYUV	29	XYUVW
4	YUVWX	64	abcde	35	bcdea
69	cdeab	98	deabc	16	eabcd
40	fghij	44	ghijf	79	hijfg
42	ijfgh	68	jfghi	13	kmnop
64	mnopk	83	nopkm	30	opkmn
33	ponmk				

This new version of Sequential STV terminates after 835 STV counts, whereas the old version would, we estimate, have required over 177,000 counts. We emphasise again that the voting pattern is highly artificial — in a real election, with 40 candidates for 9 seats, more than 60 counts would be very unusual.

### Example 3: “Woodall’s torpedo”

With 6 candidates for 2 seats, votes are

11	AC	9	ADEF	10	BC
9	BDEF	10	CA	10	CB
10	EFDA	11	FDEB		

Sequential STV elects CD even though AB form the unique Condorcet winning set. Examining why this happens, it is found that A and B are always elected by STV from any set of 3 in which they are both present, but neither A nor B is ever elected if one of them is there but not the other. Meanwhile C is always elected if present in a set of 3 except for the one set ABC. D, E and F form a Condorcet loop. CD, CE or CF would be a second Condorcet winning set if the other two of D, E and F were withdrawn.

Such a strange voting pattern is unlikely to arise in practice. It shows that Sequential STV cannot be guaranteed to find a Condorcet winning set even where one exists but it does not shake our belief that Sequential STV is a good system; it would be hard to deny that C is a worthier winner than either A or B in this example.

## 5 Acknowledgements

We thank Douglas Woodall for devising example 3, and the referee for useful comments on earlier versions of this paper.

## 6 References

- [1] I.D. Hill and Simon Gazeley, Sequential STV — a new version. *Voting matters*, 15, 13–16. 2002.
- [2] T. Nicolaus Tideman and Daniel Richardson, Better voting methods through technology: the refinement-manageability trade-off in the single transferable vote. *Public Choice*, 103, 13–34. 2000.
- [3] D.R. Woodall, Properties of preferential election rules. *Voting matters*, 3, 8–15. 1994.

# A new way to break STV ties in a special case

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## 1 Proposal

The simplest example of a particular type of tie has three votes, AB, BA, CA, for one place. The quota is 1.5, and so, under the normal rules, one candidate is selected at random for exclusion, giving the chance of election as  $2/3$  for A and  $1/3$  for B. If it is B, A will be justifiably aggrieved, and opponents of STV will argue that a random choice has given a perverse result.

A general rule to cover cases of this type would be to say that when all continuing candidates are tied (whether for exclusion or for election), they are all to be excluded, but only for the current preferences, all later preferences being unaltered. If voting is seen as a process of cutting off the top preference of each vote as soon as the fate, election or exclusion, of the candidate concerned has been decided, and reducing the value of the vote in the case of election, then this proposal introduces a new type of exclusion in which the top candidate is cut off in the normal way, but the candidate is not removed from any other votes.

The above votes, but with two places to be filled, give an example of a tie for election. Under the normal rules, whichever candidate is elected first, each of the other two has an equal chance of second place. So each of the three candidates has  $2/3$  of a chance of being elected. Under the proposals, A wins with 2 votes, B is elected with 1, and C gets none.

A possible objection is that the proposal violates the rule that later preferences must never be looked at until the fate of earlier ones has been decided, and there is a danger that it might discourage sincere voting, but this seems unlikely, and is out-weighted by its advantages if voting is sincere.

If Borda's method of counting votes is used for tie-breaking, this proposal would not be necessary; but it

has the advantage of being less of a departure from the present system.

This tie is very unlikely except in small elections, but it might well occur if partners are voting for a senior partner. If the proposal is considered too sweeping, it could be restricted to the case where the voters are the same as the candidates, and they each vote first for themselves. This would still give most of the benefits.

A powerful test of any proposed change to vote counting is, "Would it, compared with other rules, make any voters or candidates justifiably aggrieved, or lead to insincere voting?" This proposal gains on the first test, and only loses slightly on the second. Allowing parties to put up more candidates than they can hope to get in, and discouraging tactical voting, are also important, but not likely to be affected by changes in tie-breaking rules.

## 2 Editorial notes on tie breaking

The question of ties with STV has arisen several times in *Voting matters*. The previous material can be summarised as follows:

- Earl Kitchener in Issue 11 of *Voting matters* advocates the use of Borda scores [1].
- David Hill in Issue 12 argues against the use of Borda scores [2].
- Jeff O'Neill in Issue 18 notes that many rules use a first-difference rule, but he advocates a last-difference rule [3].
- Wichmann considers the use of computers in Issue 19. Here, the suggestion is that no specific rule is needed and that the computer can try all options and the result taken can be the most likely one [4].
- Earl Kitchener has returned to the subject with an alternative proposal to Borda scores in a special case which appears above.

## 2.1 Existing rules

The ERS rules [6] and the Church of England rules use the first-difference method in an attempt to break a tie.

The Meek algorithm [7] uses a deterministic algorithm based upon a random number generator to break a tie. No manual intervention is used. The New Zealand variant uses a similar method.

When the Church of England rules are applied using a computer, then the software must break the ties without manual intervention in a manner which is not defined (by the rules).

For Ireland, the manual rules are being computerized and have been used for three trial constituencies in 2002. Here, tie-breaking invokes a manual procedure, i.e. the computer software does not break the tie.

A curiosity is that in the Irish rules if when allocating surplus remainders there is a tie of the fractional part, the surplus vote is given to the candidate with the largest total number of papers from that surplus; if that is also tied then first difference is used.

It seems that a Condorcet comparison has been used to resolve a strong tie between A and B (i.e. tie can't be broken by first/last difference) in very small manual counts i.e. examine the papers to see how many times A is ahead of B compared to vice versa.

## 2.2 Discussion

This section was produced as a result of an email debate; those contributing included: James Gilmour, David Hill, Michael Hodge, Joe Otten, Joe Wadsworth and Douglas Woodall.

A number of issues arise from tie-breaking:

**Are tie-breaking rules needed?** Surely better to have a rule than toss a coin?

If a rule like first-difference, fails to break the tie, then drawing lots or some computer equivalent is needed unless we allow later preferences to be looked at. But the disadvantages remain formidable as we are then unable to promise that later choices cannot upset earlier ones. These extra tie-breaking rules complicate the counting process, since ties can arise in more than one way. It seems that just drawing lots would be adequate.

If we are saying that for:

1	AB
1	BA
1	CA

fairness demands A is elected, the same would apply to

1000	AB
1000	BA
1000	CA

So what about

999	AB
1000	BA
1000	CA

Or even

1000	AB
1001	BA
1000	CA

It seems that if the logic of looking at later preferences is sound and compelling, then they should be considered in these later examples. They are all almost identical with almost the same support for A, yet B wins with probability 1/3, 1 and 1/2 respectively. If the 1/3 should be 0, on the grounds of later preferences, perhaps the 1 or 1/2 should be reduced too?

There seems nothing in the logic of the argument that limits it to ties. Why not judge all exclusions on the basis of 'probability of election' in some sense given an analysis of all later preferences, limited only by a 'probably-later-no-harm' principle defined statistically?

This would be a rival to STV, to be considered on its merits, without muddying the waters by introducing features of it to STV for extremely marginal benefits. The claim being made here is that we want the Condorcet winner (or a similar result in the multi-seat case) rather than the AV winner. The argument is quite separate from tie breaking as such, and Condorcet-type rules need paradox breakers as well as tie breakers. If anything of the sort is to be considered, then Sequential STV [8] could be the starting point.

### If rules are used, what criteria are appropriate?

There is significant opposition to using later preferences in breaking a tie, see [2], for instance. One can argue against this on the grounds that it is hard to observe the difference between *any* tie-breaking logic and a random choice.

There was significant support for using the last-difference rule as opposed to the first-difference rule. One correspondent wrote of the latter, "It would be a bit like requiring the Speaker, in the event of a tied vote in the House, to cast his vote

not in favour of the status quo, but in favour of the outcome that more closely resembled the very earliest legislation ever passed on that question.” But it can also be argued that any such rule is arbitrary and, if it is not necessary to change, it is necessary not to change.

The first-difference rule can have the effect of giving preference to first-preference votes as opposed to transfers — this seems against the spirit of STV. With a computer, one can experiment with different procedures for breaking a tie. A reasonable criterion would be the method that most reliably resulted in the election of the candidates with the highest probabilities of being elected from breaking the ties in all the possible ways. The special case that Kitchener uses would always give the optimal result, but it is unclear how often that special case arises.

The use of Borda scores is not liked by the supporters for STV, but it is unclear if similar perverse results could be obtained if Borda scores were introduced only to break ties.

The issue of voter satisfaction has been raised. It certainly seems unsatisfactory that all the existing rules will report a random choice for elections in which the choice does not change the candidates elected. This is quite common with candidates with very low numbers of first-preferences. However, the following could be proposed to measure voter satisfaction in a tie-breaking rule:

- the method which maximizes the voters contributing to those elected;  
Maximising voters seems to accord to the *inclusive* view of STV which allows voters to be added to those supporting an already elected candidate as occurs with the Meek rules.  
The conventional approach of the manual rules is *exclusive* in which voters are not added to the list of those supporting an already elected candidate.
- the method which minimizes the non-transferable votes.  
The conventional practice with the manual rules is to minimise the non-transferable votes by considering transferable votes first when transferring surplus. In contrast, the Meek rules do not do this. However there are those who would claim that any proposal artificially to reduce non-transferables is immoral, in that it distorts what the voters have asked for.

### 3 References

- [1] Earl Kitchener. Tie-Breaking in STV. *Voting matters*. Issue 11, pp 5–6. (see <http://www.mcdougall.org.uk/VM/ISSUE11/P3.HTM>)
- [2] I. D. Hill. Tie-breaking in STV. *Voting matters*. Issue 12, pp 5–6. (see <http://www.mcdougall.org.uk/VM/ISSUE12/P2.HTM>)
- [3] J. C. O’Neill. Tie-Breaking with the Single Transferable Vote. *Voting matters*. Issue 18, pp 14–17. (see <http://www.mcdougall.org.uk/VM/ISSUE18/I18P6.PDF>)
- [4] B. A. Wichmann: Tie Breaking in STV. *Voting matters*. Issue 19, pp 1–5. (see <http://www.mcdougall.org.uk/VM/ISSUE19/I19P1.PDF>)
- [5] GS1327: General Synod, Single Transferable Vote regulations 1990 and 1998. (Obtainable from Church House Bookshop, Great Smith Street, London SW1P 3BN.)
- [6] R. A. Newland and F. S. Britton. *How to conduct an election by the Single Transferable Vote*. ERS 3rd Edition. 1997. (see <http://www.electoral-reform.org.uk/votingsystems/stvrules.htm>)
- [7] I. D. Hill, B. A. Wichmann and D. R. Woodall. Algorithm 123 — Single Transferable Vote by Meek’s method. *Computer Journal*. Vol 30, pp277–281, 1987.
- [8] I.D. Hill and Simon Gazeley. Sequential STV — a further modification. *Voting matters*, Issue 20, pp6–8.

# Apportionment and Proportionality: A Measured View

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## 1 Introduction

Collins (2003) **English Dictionary** defined ‘Proportional Representation’ (PR) as: “representation of parties in an elective body in proportion to the votes they win”. Few elections translate every Party Vote-fraction into the same Seat-fraction, thereby mediating exact PR; and raising the question of when to describe an election as full PR, semi-PR (‘broad PR’) or non-PR.

According to Gallagher, Marsh and Mitchell [11], “Ireland uses the system of proportional representation by means of the single transferable vote (PR-STV) at parliamentary, local, and European Parliament elections (the president, too is elected by the single transferable vote)”. Presidential single-member STV is **Alternative Voting** (AV), which also elects the Australian House of Representatives.

Is AV therefore a PR electoral system? The Independent Commission on the Voting System [13] — the Jenkins Report — maintained that AV alone “is capable of substantially adding to [‘First-Past-the-Post’ (FPP)] disproportionality”. The more recent Independent Commission on PR [12] affirmed that “AV can produce a hugely disproportionate result”.

How should we compare the Party disproportionality of different electoral systems? Which is the fairest — most proportional — electoral system? In other words, how should disproportionality — departure from exact PR — be quantified?

## 2 Apportionment

First consider the analogous question of the fairest method of apportionment. Collins (2003) **English Dictionary** defined ‘apportionment’ as: “*U.S. government.* the proportional distribution of the seats in a legislative

body, esp. the House of Representatives, on the basis of population”.

The USA has long wrestled with the problem of the most representative apportionment; trying various methods (Balinski and Young [1]). **Table 4.1** gives the apportionment of 105 Seats among 15 States in the first (1791) House of Representatives, applying the main five Divisor methods. For the five most and least populous States, proportionality is measured as the ratio between their aggregate Seat-fractions and Population-fractions ( $S\%/P\%$ ).

Adams, Dean and Hill yield the same apportionment: slightly *under*-representing the five most populous States ( $S\%/P\% = 0.99$ ); while *over*-representing the five least populous States ( $S\%/P\% = 1.09$ ). These methods produce a **Relative Bias** of + 10 percent (Bottom/Top third  $S\%/P\% = 1.09/0.99 = 1.10$ ).

On the other hand, Jefferson *over*-represents the top five States ( $S\%/P\% = 1.02$ ); and *under*-represents the least populous States ( $S\%/P\% = 0.89$ ): a Relative Bias of – 13 percent (Bottom/Top third  $S\%/P\% = 0.89/1.02 = 0.87 = 1 - 0.13$ ). With the lowest Relative Bias (– 2 percent), Webster yields the fairest 1791 Apportionment.

Requiring at least one Seat per State usually over-represents the least populous States. Eliminating that constraint — so quantifying method-specific bias more precisely — **Table 4.1** (bottom panel) gives the Mean Bias for all 22 USA apportionments (1791–2000). The Webster (Sainte-Laguë) Method proved the least biased overall (averaging 0.1 percent); whereas Adams (Smallest Divisor) and Jefferson (d’Hondt) were the most biased (over 20 percent).

## 3 Apportioning England

Nearer home, **Table 4.2** apportions 71 MEPs between the nine English Regions, applying the five Divisor methods to their 1999 Electorates. Adams and Dean co-

incided but, despite identical Bottom/Top third Relative Bias, differed slightly from Hill and Webster. Which apportionment is fairer?

The *European Parliament (Representation) Act 2003* prescribes that: “the ratio of electors to MEPs is as nearly as possible the same in each electoral region”. In testing fairness, the Electoral Commission [7] accepted a measure that “involves calculating for each region the difference between the number of electors per MEP for that region and the overall number of electors per MEP, and adding up all these differences (having ignored minus signs). The smaller this total is, the more equitable the outcome”.

A little mathematical notation helps here. The overall number of Electors per MEP,  $E/S = \sum E_R/S_R$ , where  $\sum$  (Sigma) denotes ‘Sum’ (over all Regions);  $E_R$  is the number of electors in a Region; and  $S_R$  is the corresponding number of seats. Each Regional deviation is the absolute difference (that is, ignoring negative signs) between its  $E_R/S_R$  and  $E/S$ ; and

$$\begin{aligned} \text{Total Deviation} &= \sum |E/S - E_R/S_R| \\ &= E/S \sum |1 - (E_R/E)/(S_R/S)| \\ &= E/S \sum |1 - E_R\%/S_R\%|, \end{aligned}$$

where  $E_R\%$  and  $S_R\%$  are the Regional Elector- and Seat- fractions (*percent*), respectively.

For any given apportionment, total Electors and Seats — and thus  $E/S$  — are fixed: hence Regional MEP apportionment is required to minimise  $\sum |1 - E_R\%/S_R\%|$ . The UK statutory criterion implies the Dean Method (Balinski and Young [1]).

Nonetheless, for the June 2004 European Elections, the Electoral Commission [7] recommended the Webster (Sainte-Laguë) Method, making the ratio of *MEPs to electors* as nearly as possible the same in each Region (beyond the statutory minimum of three MEPs). Based on December 2002 Regional electorates, Dean and Webster apportionments coincided.

We may therefore define a **Dean Index** =  $\sum |1 - E_R\%/S_R\%|$ ; and a **Webster Index** =  $\sum |1 - S_R\%/E_R\%|$ . **Table 4.2** (bottom panel) confirms that the Dean Method minimises the Dean Index; and the Webster Method minimises the Webster Index.

#### 4 Paradox and Proportionality

Overall measures of malapportionment (like the Dean and Webster indices defined above) are better than partial measures (like Bottom/Top third Relative Bias).

The Webster Method minimises total *relative* differences between Regional Elector-fractions and Seat-fractions:

$$\begin{aligned} \text{Webster Index} &= \sum |1 - S_R\%/E_R\%| \\ &= \sum |E_R\% - S_R\%|/E_R\%. \end{aligned}$$

Total *absolute* differences between Regional Elector-fractions and Seat-fractions are minimised by the **Hamilton Method** (Largest Remainders: LR–Hare).

This Quota Method allocates to each Region the integer part of its proportional entitlement (number of Hare Quotas: one Hare Quota = National Electors/National Seats). Any residual seats are then allocated to the regions with the largest fractional parts (remainders) of Hare Quotas.

We may therefore define a **Hamilton Index** =  $\sum |E_R\% - S_R\%|$ ; minimised by the Hamilton Method. Applied to all 22 USA apportionments (without seat minima), Hamilton averages a (Bottom/Top third) Relative Bias of –0.3 percent: differing insignificantly from Webster (–0.1 percent).

Unlike Webster, the Hamilton Method of apportionment is vulnerable to paradox: notably the Alabama Paradox. The 1880 USA Census disclosed that, if total House size were *increased* from 299 to 300 seats, then the Hamilton apportionment to Alabama would have *decreased* from eight to seven seats (Balinski and Young [1])!

That consideration excludes Hamilton as a method of apportionment; though not necessarily for evaluating malapportionment. So how best to quantify malapportionment — or disproportionality?

#### 5 Party Disproportionality

Gallagher [10] concluded that each PR method “minimizes disproportionality according to the way it defines disproportionality”. However, Lijphart [14] argued that LR-Hare (Hamilton) and Sainte-Laguë (Webster) mediate “inherently greater proportionality” than d’Hondt (Jefferson); thereby justifying proportionality measures “biased in favour of LR-Hare”.

LR-Hare minimises the **Loosemore-Hanby Index** (Loosemore and Hanby, [15]):

$$\text{LHI (percent)} = \frac{1}{2} \sum |V_P\% - S_P\%|,$$

where  $V_P\%$ ,  $S_P\%$  = Party Vote-, Seat-fractions (*percent*).

Compare the Hamilton Index  $= \sum |E_R\% - S_R\%|$ , as defined above. Halving the sum ensures that LHI ranges 0–100 percent.

LHI is the ‘DV score’ mentioned by the Independent Commission on the Voting System [13]; and as defined by the Independent Commission on PR [12]. LHI complements the **Rose Proportionality Index** (Mackie and Rose, [16]) *percent*:

$$= 100 - \frac{1}{2} \sum |V_P\% - S_P\%| = 100 - \text{LHI} (\text{percent}).$$

**Table 4.3** illustrates the calculation of LHI and RPI for the 2004 European Parliamentary Election in Britain. Over-represented and under-represented Party Total Deviations are necessarily equal and opposite ( $\pm 14.7$  percent in **Table 4.3**); and Party total over-representation is simply the Loosemore-Hanby Index (LHI = 14.7 percent).

## 6 Debate

As a measure of Party disproportionality, the Loosemore-Hanby Index (LHI) has been criticised on three main grounds: for violating Dalton’s Transfer Principle (Taagepera and Shugart [22]); for being vulnerable to paradox (Gallagher [10]); and for exaggerating the disproportionality of PR systems involving many parties (Lijphart [14]).

Dalton’s Transfer Principle states that transferring wealth from a richer to a poorer person decreases inequality, decreasing any inequality index (Taagepera and Shugart [22]). However, transferring seats between over-represented parties (or between under-represented parties) leaves LHI unchanged.

Thus in the 2004 European Election in Britain (**Table 4.3**), imagine the Conservatives (from 27 to 25 seats) losing two seats to Labour (from 19 to 21 seats). Then both Party deviations would converge ( $S_P\% - V_P\% =$  from + 9.3 to + 6.6 percent, and from + 2.7 to + 5.4 percent, respectively); decreasing GhI (from 8.3 to 7.7 percent), leaving LHI unchanged (14.7 percent). However, Party total over-representation remains unchanged: so why should overall disproportionality change?

Minimised by LR-Hare (Hamilton), LHI is susceptible to the paradoxes of that Quota method (Gallagher [10]). Because Sainte-Laguë (Webster) is the least biased Divisor method — and immune to paradox — Gallagher [10] recommended a **Sainte-Laguë Index** “as the standard measure of disproportionality”:

$$\text{SLI} (\text{percent}) = \sum (V_P\% - S_P\%)^2 / V_P\%.$$

However, in a single-member constituency, if the winner receives under half of all votes, then SLI exceeds 100 percent (unlike LHI, which measures unrepresented — wasted — votes).

Nowadays, Gallagher [10] is mainly cited for his ‘Least Squares Index’:

$$\text{GhI} (\text{percent}) = \sqrt{\frac{1}{2} \sum (V_P\% - S_P\%)^2}.$$

Also minimised by LR-Hare, GhI is subject to the same paradoxes as LHI. Gallagher [10] saw GhI as “a happy medium” between LHI and the **Rae Index** (Rae [18]):

$$\text{RaI} (\text{percent}) = \sum |V_P\% - S_P\%| / N,$$

where  $N =$  Number of parties ( $V_P\% > 0.5$  percent).

Thus RaI measures *average* deviation per Party; whereas LHI measures (half) *Total Deviation*. Yet why hybridise such conceptually distinct measures in one measure (GhI)?

Taagepera and Grofman [21] have attributed the recent shift, from LHI towards GhI, “to sensitivity to party system concentration”; based on the intuition of Lijphart [14] that a few large deviations ( $V_P\% - S_P\%$ ) should be evaluated as more disproportional overall than many small deviations with the same Total Deviation (and hence LHI). It remains unclear why larger Party deviations should be potentiated; and smaller ones attenuated.

For example, in the 2004 European Election in Britain, exact GhI was 8.3 percent. However, aggregating unrepresented parties ( $S_P\% = 0.0$  percent: **Table 4.3**) increases GhI to 10.7 percent; leaving LHI unchanged (14.7 percent). In the process, Party total under-representation has not changed: so why should Total Disproportionality change? Likewise, in single-member constituencies, GhI depends on the division of votes among losing candidates.

Monroe [17] proposed an inequity index rather similar to GhI:

$$\text{MrI} (\text{percent}) = \sqrt{\frac{\sum (V_P\% - S_P\%)^2}{1 + \sum (V_P\%/100)^2}}$$

LR-Hare also minimises MrI; which falls below 100 percent for extreme disproportionality involving more than two parties (like GhI, but unlike LHI).

Taagepera and Shugart [22] mentioned an electoral analogue of the widespread Gini Inequality Index, with several examples; but without defining any **Gini Disproportionality Index** (GnI). It turns out that **GnI** (*percent*):

$$= \sum \sum | (V_P\% \times S_Q\%) - (S_P\% \times V_Q\%) | / 200$$

Thus GnI sums the absolute differences between the  $S_P\%/V_P\%$  ratios of every pair of parties, weighted by the product of their vote-fractions ( $V_P\%/100$ ). This complex GnI satisfies Dalton’s Transfer Principle; and aggregating unrepresented parties ( $S_P\% = 0.0$  percent) leaves GnI unchanged (like LHI and SLI).

Taagepera and Grofman [21] evaluated 19 disproportionality indices against 12 criteria, sustaining five measures: LHI; GhI; SLI (‘chi-square’); MrI; and GnI. Nonetheless, they overlooked both a Farina Index (FrI) and a Borooah Index (BrI).

Woodall [24] cited JEG Farina for a vector-based measure of Party Total Disproportionality: the angle between two multidimensional vectors, whose coordinates are Party vote and seat numbers. Its fraction of a right angle defines a **Farina Index**, FrI (*percent*) =

$$\arccos \left[ \frac{\sum (S_P\% \times V_P\%)}{\sqrt{\sum S_P\%^2 \times \sum V_P\%^2}} \right] \times 100/90^\circ$$

ranging 0–100 percent (instead of 0–90 degrees).

Borooah [2] proposed an electoral analogue of the Atkinson Inequality Index, depending on “society’s aversion to inequality” (like Gini, originally measuring income inequality). Establishing national ‘Societal Aversion to Disproportionality’ seems arbitrary; while a moderate value ( $SAD = 2$ ) defines a **Borooah Index**,

BrI (*percent*) =  $100 - 1/[\sum (S_P\%/100)^2/V_P\%]$ , ranging 0–100 percent.

## 7 Correlations

For 82 general elections in 23 countries (1979–89), Gallagher [10] reported high correlations between LHI, GhI and SLI. Graphing high correlations between LHI, GhI, SLI and FrI, Wichmann [23] noted that central placement reinforced LHI.

**Table 4.4** gives the correlations between all seven indices in the last 44 UK general elections (1832–2005). Most notably, LHI proved very highly correlated with GnI; GhI with MrI; and SLI with BrI ( $R > 0.99$ ). Indeed, LHI and GnI were highly correlated ( $R > 0.95$ ) with all other measures of Party Total Disproportionality.

## 8 Proportionality Criteria

The Independent Commission on the Voting System [13] observed that “full proportionality ... is generally considered to be achieved as fully as is normally practicable if [LHI%] falls in the range of 4 to 8”. More generously, we might allow LHI under 10 percent to characterise **full PR**. LHI ranging 10–15 percent could then encompass **semi-PR** (‘broad PR’); with LHI over 15 percent constituting **non-PR**.

In **UK** general elections (FPP) since World War I, LHIs have ranged from 27 percent (1918); to only four percent (1951) — ironically, when the Conservatives won fewer votes, but more seats, than Labour (Rallings and Thrasher [19]). In the last nine general elections (1974–2005: **Table 4.5**), LHIs have ranged 15–24 percent, averaging 20 percent: clearly *non-PR*.

What of the nominally PR elections, introduced in Britain since 1997? In the 1999 and 2004 European Parliamentary elections, Regional d’Hondt yielded LHIs of 14.1–14.7 percent (between Party List votes and MEPs) nationwide: barely *semi-PR*. Likewise applied regionally, either Sainte-Laguë (LHI = 6.1–8.4 percent), or LR–Hare (LHI = 6.1–5.4 percent), would have mediated *full PR*. So the method used here can make a considerable difference.

In the 1999 and 2003 **Scottish Parliament** and **National Assembly for Wales** elections, between Party List votes and Total (FPP Constituency + Additional Regional) seats, LHIs ranged 11–14 percent. The 2000 and 2004 **London Assembly** elections (also FPP-plus, but with a five percent Party Vote Threshold) yielded similar Party List LHIs of 14–15 percent. Thus all three British Regional Assemblies remain *semi-PR* at best (Independent Commission on PR [12]).

In contrast, both 1998 and 2003 **Northern Ireland Assembly** elections (multi-member STV) mediated First Preference LHIs of only 6.0–6.4 percent: *full PR*. **Table 4.6** ranks UK national and regional election LHIs over the past decade (1995–2005).

## 9 Vote Transferability and District Magnitude

Transferable voting complicates evaluating the disproportionality of both AV and multi-member STV. First Count LHI is *not* the sole criterion; though Final Count LHI over-estimates Party proportionality (Gallagher [9]). For comparing transferable voting with

other electoral systems, averaging First and Final Count LHIs appears reasonable.

Under Alternative Voting (AV), in the last nine general elections in **Australia** (1983–2004), First Count LHI ranged 12–20 percent, averaging 16 percent (**Table 4.5**): practically *non-PR*. Final Count LHI ranged 5–13 percent, averaging eight percent (PR); while mean First + Final Count LHI averaged 12 percent: *semi-PR* overall (compare **Table 4.6**).

So much for empirical claims that AV “is capable of substantially adding to [FPP] disproportionality” (Independent Commission on the Voting System, [13]). FPP votes — involving tactical considerations — should not only be compared with AV First Preferences.

Taagepera and Shugart [22] called AV ‘semi-PR’; and attributed any ‘semi-PR effect’ in multi-member STV elections to low **District Magnitude** ( $M = \text{Number of Seats per Constituency}$ ). As Gallagher [9] noted: “the smaller the constituency [M], the greater the potential for disproportionality”; and reported decreasing LHI with increasing STV District Magnitude in 16 Irish general elections (1927–1973).

**Table 4.7** gives national aggregate LHI, by District Magnitude and Count, in the last 13 Irish general elections (1961–2002). Between such low District Magnitudes ( $M = 3\text{--}5$ ), disproportionality might be expected to fall steeply: so the relative insignificance of all LHI differences is remarkable.

Overall, First Count LHIs ranged 3–13 percent (averaging seven percent); Final Count LHIs ranged 1–7 percent (averaging three percent); and mean First + Final Count LHI averaged only five percent (and 6–7 percent for  $M = 3\text{--}5$ ). Virtually regardless of District Magnitude, multi-member STV mediates *full PR*.

## 10 Conclusions

Sainte-Laguë (Webster) is the most equitable method of apportionment — and the most proportional electoral principle. The d’Hondt (Jefferson) Method over-represents the most populous regions (and the most popular parties).

Not much has changed since Gallagher [10] lamented “surprisingly little discussion of what exactly we mean by proportionality and how we should measure it”. Certainly, Party disproportionality indices have proliferated; among which the Loosemore-Hanby Index (LHI) — straightforwardly measuring Party total over-representation — remains the most serviceable. More-

over, such absolute disproportionality is what matters politically [14, 21].

Continuing debate on the ‘best’ measure of disproportionality may distract attention from the main task of evaluating the relative disproportionality of different electoral systems. Taagepera and Grofman [21] marginally preferred the Gallagher Index (GhI); allowing that its advantages over LHI were debatable.

LHI fails Dalton’s Transfer Principle; yet transferring seats between over- (or under-) represented parties should arguably not change a measure of Total Disproportionality. LHI, GhI and MrI alike remain vulnerable to the paradoxes of the Largest Remainders (LR-Hare/Hamilton) Method.

The Sainte-Laguë Index (SLI) is unsuitable for measuring Party Total Disproportionality. Fortunately highly correlated with LHI, the Gini Disproportionality Index (GnI) is rather complicated to explain and calculate (virtually necessitating computerisation). Interestingly, Riedwyl and Steiner [20] traced the LHI concept back to Gini (1914–15).

Settling for the most elementary LHI clearly demonstrates that, in recent UK general elections, FPP has proved *non-PR*. Even nominally PR elections in Britain have barely mediated *semi-PR*. Yet in both Northern Ireland Assembly elections, multi-member Single Transferable Voting has yielded *full PR* of Party First Preferences.

Allowing for vote transferability, STV has also mediated full PR in recent Irish general elections; hardly affected by District Magnitude (between three and five seats per constituency). Likewise in Australia, Alternative Voting has arguably proved semi-PR; and certainly no more disproportional than First-Past-the-Post.

## 11 References

- [1] Balinski, M. and Young, H. P. (2001): *Fair Representation*. Meeting the Ideal of One Man, One Vote. Brookings Institution Press, Washington DC.
- [2] Borooah, V. K. (2002): ‘The Proportionality of Electoral Systems: Electoral Welfare and Electoral Inequality’: *Economics and Politics* **14**, 83–98.
- [3] Chief Electoral Officer for Northern Ireland (1998): *The New Northern Ireland Assembly Election 25 June 1998: Election of 6 Members to*

- each of the 18 (NI) Westminster Constituencies. Belfast.
- [4] Dáil Éireann (1962–2003): *Election Results and Transfer of Votes in General Election: October 1961 – May 2002*. Stationery Office, Dublin.
- [5] Electoral Commission (2001): *Election 2001: The Official Results*. London.
- [6] Electoral Commission (2003a): *Combination of Gibraltar with a European electoral region in England and Wales*. Consultation paper, June 2003.
- [7] Electoral Commission (2003b): *Distribution between electoral regions of UK MEPs*. Recommendation, October 2003.
- [8] Electoral Office for Northern Ireland (2004): *Northern Ireland Assembly Elections 26 November 2003 Results*.
- [9] Gallagher, M. (1986): 'The Political Consequences of the Electoral System in the Republic of Ireland': *Electoral Studies* 5, 253–75.
- [10] Gallagher, M. (1991): 'Proportionality, Disproportionality and Electoral Systems': *Electoral Studies* 10, 33–51.
- [11] Gallagher, M, Marsh, M. and Mitchell, P. eds (2003): *How Ireland Voted 2002*. Palgrave Macmillan, Basingstoke.
- [12] Independent Commission on PR (2004): *Changed Voting Changed Politics*. Lessons of Britain's Experience of PR since 1997. Constitution Unit, School of Public Policy, UCL, London.
- [13] Independent Commission on the Voting System (1998): *The Report of the Independent Commission on the Voting System*. Cm 4090–I. The Stationery Office, London.
- [14] Lijphart, A. (1994): *Electoral Systems and Party Systems*. Oxford University Press.
- [15] Loosemore, J. and Hanby, V. (1971): 'The Theoretical Limits of Maximum Distortion: Some Analytic Expressions for Electoral Systems': *British Journal of Political Science* 1, 467–77.
- [16] Mackie, T. T. and Rose, R. (1991): *The International Almanac of Electoral History*. Macmillan, Basingstoke.
- [17] Monroe, B. L. (1994): 'Disproportionality and Malapportionment: Measuring Electoral Inequity': *Electoral Studies* 13, 132–49.
- [18] Rae, D. (1967): *The Political Consequences of Electoral Laws*. Yale University Press, New Haven.
- [19] Rallings, C. and Thrasher, M. eds (2000): *British Electoral Facts 1832–1999*. Ashgate, Aldershot.
- [20] Riedwyl, H. and Steiner, J. (1995): 'What is Proportionality Anyway?': *Comparative Politics* 27, 357–369.
- [21] Taagepera, R and Grofman, B (2003): 'Mapping the Indices of Seats-Votes Disproportionality and Inter-Election Volatility': *Party Politics* 9, 659–677.
- [22] Taagepera, R. and Shugart, M. S. (1989): *Seats and Votes*. The Effects and Determinants of Electoral Systems. Yale University Press, New Haven.
- [23] Wichmann, B. A. (2003): 'Proportionality Revisited': *Voting matters* Issue 16, 6–8.
- [24] Woodall, D. R. (1986): 'How Proportional is Proportional Representation? *The Mathematical Intelligencer* 8 : 4, 36–46.

Table 4.1: State Population, Seat Apportionment and Relative Bias

(**Bottom/Top third** most populous States), by Divisor Method: House of Representatives, **USA: 1791** Apportionment; and 1791–2000 **Mean Bias** (22 Apportionments, without seat minima).

State of Union	Population ( <i>P</i> )	Divisor Method: Number of Seats ( <i>S</i> )				
		Adams	Dean	Hill	Webster	Jefferson
<b>Total (USA)</b>	3,615,920	105	105	105	105	105
Virginia	630,560	18	18	18	18	19
Massachusetts	475,327	14	14	14	14	14
Pennsylvania	432,879	12	12	12	13	13
North Carolina	353,523	10	10	10	10	10
New York	331,589	10	10	10	10	10
Maryland	278,514	8	8	8	8	8
Connecticut	236,841	7	7	7	7	7
South Carolina	206,236	6	6	6	6	6
New Jersey	179,570	5	5	5	5	5
New Hampshire	141,822	4	4	4	4	4
Vermont	85,533	3	3	3	2	2
Georgia	70,835	2	2	2	2	2
Kentucky	68,705	2	2	2	2	2
Rhode Island	68,446	2	2	2	2	2
Delaware	55,540	2	2	2	2	1
Top third (5)	2,223,878	64	64	64	65	66
Bottom third (5)	349,059	11	11	11	10	9
Seat/ Population fraction ( <i>S%/P%</i> )	Top third	0.99	0.99	0.99	1.01	1.02
	Bottom third	1.09	1.09	1.09	0.99	0.89
1791 <b>Relative Bias</b> , <i>percent</i> *		+10	+10	+10	-2	-13
1791–2000 <b>Mean Bias</b> , <i>percent</i> *		+20.3	+7.0	+ 5.0	-0.1	-20.7

\* **Relative Bias:** Percentage deviation from unity of ratio between Seat/Population (or *S%/P%*) ratios of **Bottom/Top third** most populous States.

**Data Source:** Balinski and Young [1].

Table 4.2: Regional Electors, Seat Apportionment and Relative Bias

(**Bottom/Top third** most populous Regions) and **Malapportionment Index**, by Divisor Method: MEPs, **England, 1999.**

Region	Electors ( <i>E</i> )	Divisor Method: Number of Seats ( <i>S</i> )				
		Adams	Dean	Hill	Webster	Jefferson
<b>Total</b> (England)	37,079,720	71	71	71	71	71
South East	6,023,991	11	11	12	12	12
North West	5,240,321	10	10	10	10	10
London	4,972,495	10	10	9	9	10
Eastern	4,067,524	8	8	8	8	8
West Midlands	4,034,992	8	8	8	8	8
Yorkshire & Humber	3,795,388	7	7	7	7	7
South West	3,775,332	7	7	7	7	7
East Midlands	3,199,711	6	6	6	6	6
North East	1,969,966	4	4	4	4	3
Top third (3)	16,236,807	31	31	31	31	32
Bottom third (3)	8,945,009	17	17	17	17	16
Seat-/Electorate- fraction ( <i>S</i> %/ <i>E</i> %)	Top third	0.997	0.997	0.997	0.997	1.029
	Bottom third	0.993	0.993	0.993	0.993	0.934
<b>Relative Bias, percent</b> *		-0.46		-0.46		-9.24
<b>Malapportionment</b>	Dean	30.96		30.98		50.01
<b>Index (percent)</b> †	Webster	31.22		31.07		45.05

\* **Relative Bias:** Percentage deviation from unity of ratio between Seat/Electorate (or *S*%/*E*%) ratios of Regions with **Bottom/Top third** most electors.

† **Malapportionment Index:**

Dean Index (percent) =  $\sum |1 - E_R\%/S_R\%| \times 100$  ; and

Webster Index (percent) =  $\sum |1 - S_R\%/E_R\%| \times 100$  :

**Data Source:** Electoral Commission [6].

Table 4.3: Analysis of Party Votes and Seats

Number, Fraction and **Loosemore-Hanby Index**: European Election (d'Hondt Regional Closed Party Lists):  
**Britain, June 2004.**

Party	Number		Fraction, <i>percent</i>		Seat–Vote Fraction
	Votes ( $V_P$ )	Seats ( $S_P$ )	Votes ( $V_P\%$ )	Seats ( $S_P\%$ )	<b>Deviation, <i>percent</i></b> ( $S_P\% - V_P\%$ ) *
<b>Total (Britain)</b>	16,448,605	75	100.0	100.0	0.0
Conservative	4,397,090	27	26.7	36.0	+9.3
Labour	3,718,683	19	22.6	25.3	+2.7
UK Independence	2,650,768	12	16.1	16.0	-0.1
Liberal Democrat	2,452,327	12	14.9	16.0	+1.1
Green	1,028,283	2	6.3	2.7	-3.6
Scottish National	231,505	2	1.4	2.7	+1.3
Plaid Cymru	159,888	1	1.0	1.3	+0.4
Others (unrepresented)	1,810,061	0	11.0	0.0	-11.0
Over-represented *	10,959,493	61	66.6	81.3	+14.7†
Under-represented	5,489,112	14	33.4	18.7	-14.7

\* Over-represented Party  $S_P\% > V_P\%$  (under-represented  $S_P\% < V_P\%$ ).

† **Loosemore-Hanby Index (LHI)** = Party total over-representation  
 $= \frac{1}{2} \sum |V_P\% - S_P\%| = \mathbf{14.7 \text{ percent}}$ .

**Rose Proportionality Index (RPI)** = Complement of Party total over-representation =  $100.0 - 14.7 = \mathbf{85.3 \text{ percent}}$ .

**Data Source: Guardian, 16 June 2004.**

Table 4.4: Correlations between Seven Party Total Disproportionality Indices

**UK (FPP: 44 general elections), 1832–2005.**  
 Values as percentages.

Index	LHI	GhI	SLI	MrI	GnI	FrI	BrI
LHI	–	96.4	91.0	97.7	98.1	96.5	91.4
GhI		–	84.8	99.8	94.0	96.4	86.1
SLI			–	86.4	92.3	84.7	99.5
MrI				–	95.4	97.2	87.5
GnI					–	94.7	93.0
FrI						–	85.6
Mean Index	11.5	9.2	11.4	11.2	13.4	11.8	9.6

**Data sources:** Electoral Commission [5]; Rallings and Thrasher [19] and Guardian, 7 May 2005.

Table 4.5: Loosemore-Hanby Index

Last Nine General Elections in **UK** (FPP), 1974–2005;  
and **Australia** (AV), 1983–2004.

<b>UK :</b> Election	<b>FPP:</b> <b>LHI, percent</b>	<b>Australia:</b> Election	<b>AV Count: LHI, percent</b>	
			First	Final *
Feb 1974	19.9	1983	15.2	11.2
Oct 1974	19.0	1984	11.8	7.9
1979	15.3	1987	13.6	9.8
1983	24.2	1990	17.1	5.0
1987	20.9	1993	14.1	7.4
1992	18.0	1996	18.8	12.6
1997	21.2	1998	20.5	6.4
2001	22.1	2001	18.2	4.9
2005	20.7	2004	15.8	6.6
1974–2005 Mean	20.1	1983–2004 Mean ( <b>First + Final</b> )	16.1	8.0 ( <b>12.0</b> )

\* AV Final Count: Two-Candidate Preferred (excluding few non-transferable votes: in Australia, valid voting necessitates rank-ordering all AV preferences).

**Data Sources:** Rallings and Thrasher [19]; Electoral Commission [5]; and Australian Electoral Commission (personal communications, 1988–2005).

Table 4.6: Loosemore-Hanby Index

By Assembly, Electoral System and Election (Year): **UK**, 1995–2005.

Assembly	Electoral System	Year	LHI, percent
House of Commons (UK MPs)	<b>FPP</b> (First-Past-the-Post)	2001	22.1
		2005	20.7
European Parliament (British MEPs)	<b>CPL</b> (Closed Party List: Regional d'Hondt)	1999	14.1
		2004	14.7
London Assembly	<b>FPP + 44% CPL</b> (Party List $V_P\% > 5\%$ )	2000	14.8
		2004	13.6
National Assembly for Wales	<b>FPP + 33% CPL</b> (Regional d'Hondt)	1999	11.2
		2003	14.1
Scottish Parliament	<b>FPP + 43% CPL</b> (Regional d'Hondt)	1999	10.5
		2003	12.5
Northern Ireland Assembly	<b>STV</b> (Six Seats per Constituency)	1998	6.0 to 3.8*
		2003	6.4 to 5.4*

\* First to Final count (excluding non-transferable votes).

**Data Sources:** Chief Electoral Officer for Northern Ireland [3]; Electoral Commission [5]; Electoral Office for Northern Ireland [8]; Rallings and Thrasher [19]; **Guardian**, 6 May 2000, 3 May 2003 and 7 May 2005; **Times**, 12 June 2004.

Table 4.7: National Aggregate Loosemore-Hanby Index

By STV District Magnitude, Count and Election:  
**Irish Republic, 1961–2002.**

Election Year (Month)	District Magnitude (Seats per STV Constituency): LHI, percent ( First to Final Count* )			
	Total	3	4	5
1961	8.4 to 3.4	9.4 to 4.5	10.7 to 7.1	9.7 to 4.7
1965	3.2 to 2.3	3.2 to 2.0	6.0 to 5.8	4.2 to 2.1
1969	7.1 to 4.5	7.3 to 4.6	7.5 to 4.5	4.3 to 2.0
1973	4.3 to 1.2	4.5 to 2.4	4.6 to 2.6	7.3 to 8.9
1977	7.4 to 4.1	7.3 to 6.0	9.7 to 4.1	8.5 to 1.1
1981	5.8 to 2.4	4.6 to 2.3	10.2 to 2.6	5.3 to 4.0
1982 (Feb)	3.4 to 1.9	2.6 to 2.0	4.4 to 2.8	4.2 to 1.1
1982 (Nov)	4.2 to 1.9	2.6 to 3.8	7.2 to 3.0	4.7 to 3.4
1987	9.9 to 1.3	10.5 to 7.3	10.9 to 2.8	10.1 to 2.2
1989	7.1 to 2.4	6.0 to 3.9	8.9 to 2.5	7.8 to 2.6
1992	8.2 to 3.7	9.8 to 3.6	10.5 to 5.6	8.5 to 3.9
1997	12.9 to 5.1	14.9 to 6.9	16.2 to 6.7	13.2 to 5.7
2002	12.6 to 6.6	15.8 to 10.4	14.2 to 6.3	11.4 to 5.6
1961–2002 Mean (First + Final)	7.3 to 3.1 (5.2)	7.6 to 4.6 (6.1)	9.3 to 4.3 (6.8)	7.6 to 3.6 (5.6)

\* Final Count: Excluding non-transferable votes.

**Data source:** Dáil Éireann (1962–2003).